Polynomial Division

Use the example below and review the algorithm for long division:

$$(x^4 - x^3 + 7x^2 + 5) \div (x^2 + 1)$$

$$
\begin{array}{r}
x^2 - x + 6 \\
x^2+1 \,\big)\, \overline{x^4 - x^3 + 7x^2 + 5} \\
x^4 + x^2 \\
\hline
-x^3 + 6x^2 + 5 \\
-x^3 - x \\
\hline
6x^2 + x + 5 \\
6x^2 + 6 \\
\hline
x - 1
\end{array}
$$

so

$$\underbrace{x^4 - x^3 + 7x^2 + 5}_{p(x)} = \underbrace{(x^2+1)}_{d(x)}\underbrace{(x^2 - x + 6)}_{g(x)} + \underbrace{x-1}_{r(x)}$$

$d(x) \uparrow$ divisor, $g(x) \uparrow$ quotient, $r(x) \uparrow$ remainder

$$p(x) = d(x)g(x) + r(x)$$

$$\frac{p(x)}{d(x)} = g(x) + \frac{r(x)}{d(x)}$$

Division of Polynomials:

Let $p(x)$ and $d(x)$ be polynomials such that $d(x) \neq 0$. Then there are unique polynomials $q(x)$ and $r(x)$, called the quotient and the remainder, respectively, such that

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

or

$$p(x) = q(x)\,d(x) + r(x)$$

If $r(x) = 0$ then q and p are factors of the polynomial p

Use the rational number $\frac{212}{3}$ and division to understand the concepts of dividend, divisor, quotient, and remainder.

$$\begin{array}{r} 70 \\ 3\,\overline{)\,212} \\ \underline{21} \\ 2 \end{array}$$

$212 = 3(70) + 2$, $q = 70$

$r = 2$

$a = bq + r$

$\frac{a}{b} = q + \frac{r}{b}$ $(r<b)$

Linear Factor Theorem:
The number k is a factor of a polynomial $p(x)$ if and only if the linear polynomial $x - k$ is a factor of p. That means, if k is a zero of $p(x)$ then:

1) $p(x) = (x - k)\, q(x)$ for some polynomial q.
2) $p(k) = 0$
3) k is an x-intercept of the graph of p.
4) If $p(x)$ is divided by $(x - k)$ then the remainder is $p(k)$.

Example: Show that 2 is a zero of the function $f(x) = x^3 - 2x - 4$
What other things can be determined about the function f based on this information?

$$f(2) = 8 - 4 - 4 = 0$$

$$x^3 - 2x - 4 = (x-2)(x^2 + 2x + 2)$$

Example: Show that $2 + \sqrt{3}$ is a zero of the function
$f(x) = x^2 - 4x + 1$
What other things can be determined about the function f based on this information?

$$f(2+\sqrt{3}) = (2+\sqrt{3})^2 - 4(2+\sqrt{3}) + 1$$

$$= 4 + 4\sqrt{3} + 3 - 8 - 4\sqrt{3} + 1$$

$$= 7 - 8 + 1 = 0$$

$$x^2 - 4x + 1 = (x - 2 - \sqrt{3})(x - 2 + \sqrt{3})$$

Review synthetic division by dividing the following:

$$\frac{3x^3 - 2x^2 + 2}{x + 2}$$

$$\begin{array}{r|rrrr} -2 & 3 & -2 & 0 & 2 \\ & \downarrow & -6 & 16 & -32 \\ \hline & 3 & -8 & 16 & -30 \end{array}$$

$$3x^3 - 2x^2 + 2 = (x+2)(3x^2 - 8x + 16) - 30$$

If $p(x) = 3x^3 - 2x^2 + 2$, $p(-2) = -30$

Divide: $(2x^5 + x^4 - 18x^3 - 9x^2 + 16x + 8) \div (x + \frac{1}{2})$

$$\begin{array}{r|rrrrrr} -\frac{1}{2} & 2 & 1 & -18 & -9 & +16 & +8 \\ & & -1 & 0 & +9 & 0 & -8 \\ \hline & 2 & 0 & -18 & 0 & 16 & 0 \end{array}$$

$$p(x) = 2x^5 + x^4 - 18x^3 - 9x^2 + 16x + 8$$

$$= (x + \tfrac{1}{2})(2x^4 - 18x^2 + 16)$$

$$p(-\tfrac{1}{2}) = 0$$